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Unsteady expansions into vacuum with spherical symmetry

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In this paper time dependent expansions of monatomic gases with spherical symmetry are discussed. For the particular case of Maxwellian molecules closed expressions for the moments up to second order are obtained in regions of the flow where the inviscid solution is no longer valid. These solutions are derived in a general form using the particle path function as a parameter. The structure of the inviscid solution is such that this simplification can be made. The novelty of the present approach is that solutions already derived in previous papers can be obtained from the general solution in various limits; both the results for steady flow and the expansion of a fixed mass of gas can be derived in this manner. Finally, a particular example is constructed in order to illustrate the general theory.

1. Introduction

In this paper the unsteady expansion of a monatomic gas into vacuum for flows with spherical symmetry is studied from the viewpoint of the kinetic theory of gases. The problem of the expansion of such gases has been studied in detail recently but attention has hitherto been confined to purely steady or unsteady flows. The possibility of an unsteady flow approaching the steady state is considered below in general form with the intention of unifying previous theoretical approaches to the problem.

The steady spherically symmetric problem has been considered by various authors (Edwards & Cheng 1966; Freeman 1967; Freeman & Thomas 1969; Hamel & Willis 1966). As far as this paper is concerned the work of Freeman (1967) is of importance. The techniques developed by him form the basis on which the present paper is written, namely, the application of the method of matched asymptotic expansions to this type of problem. The theory presented below is based to a large extent on the use of particle path co-ordinates. It is shown that in this co-ordinate system, the outer zeroth-order solution can be treated as though each particle path line were completely independent, a simplification exploited previously in the purely unsteady problems discussed by Freeman & Grundy (1968) and Grundy & Thomas (1969). In the present problem the steady flow limit is extracted in certain regions of the flow field where the asymptotic velocity becomes independent of particle path line.

Throughout this work the Maxwell molecule collision model is used in order that closed moment equations can be obtained. This is a necessary mathematical restriction and corresponds to an inverse fifth power law for the molecular interaction field.

2. The inviscid solution

The basic gasdynamic problem of the unsteady expansion of a gas into vacuum from the viewpoint of continuum thermodynamics has been treated most extensively in the literature (Stanyukovitch 1960; Courant & Friedrichs 1948). Exact analytical solutions to the inviscid equations for particular symmetries and boundary conditions have only been found in a very restricted number of cases. First, there is the classical problem of the expansion of a semi-infinite mass of a uniform gas into vacuum, the flow field comprising a simple wave and and a uniform state. Secondly, there is the plane expansion of a uniform finite mass of gas which possesses a simple wave solution, a constant state and an interaction region in which there is an analytical solution. A further class of exact solutions can be obtained for spherical, cylindrical and plane flows; these are self-similar solutions which have specified initial conditions (Thornhill 1958 and Sedov 1959).

The existence of these exact solutions has led investigators into attempting to find solutions which are usually valid in some asymptotic sense. In particular for expanding flows where density is a decreasing function of time (or space co-ordinate), large time (or distance) solutions exhibit simplifications which are particularly useful in examining non-equilibrium effects. However, it is an inherent drawback of these limiting solutions that they are in a sense indeterminate, due to the loss of the initial conditions governing the flow which, of course, cannot be applied. The analysis of this section will be concerned with inviscid spherically and cylindrically symmetric flows, the similarity between the two geometries makes it convenient to consider both cases simultaneously. Large distance expansions of the thermodynamic variables will be generated and the corresponding zeroth-order terms evaluated.

The inviscid equations of motion for the problem can be written,

$$\left.\begin{aligned} \frac{\partial(nr^\sigma)}{\partial t}+\frac{\partial}{\partial r}(nur^\sigma) &= 0,\\ \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+n^{\gamma-2}\frac{\partial n}{\partial r} &= 0,\\ p &= nT, \end{aligned}\right\} \tag{2.1}$$

where the relation $p = n^\gamma$ has been used to eliminate pressure from the momentum equation, γ is the ratio of specific heats and σ is the dimension index, $\sigma = 1$ for cylindrical flow and 2 for spherical flow. The variables have been non-dimensionalized

$$\left.\begin{gathered} n = \frac{n'}{n_0'},\quad u = \frac{u'}{a_0'},\quad r = \frac{r'}{L},\quad t = \frac{t'a_0'}{L},\\ p = \frac{p'}{cn_0'^{\gamma}},\quad T = \frac{RT'}{cn_0'^{\gamma-1}}, \end{gathered}\right\} \tag{2.2}$$

where n_0' and a_0' are the number density and sound speed at $r = L$, $t = 0$. L is a characteristic radius at which the flow is in a continuum state, R is the gas constant and c is the constant occurring in the relation $p = cn^\gamma$, the primes referring to dimensional variables. T, p, n and u are therefore the non-dimensional temperature, pressure, density and mean gas velocity.

A new variable ψ, the particle path function, is defined by

$$\left.\begin{aligned} nr^\sigma &= \frac{\partial \psi}{\partial r} \\ \text{and} \qquad nur^\sigma &= -\frac{\partial \psi}{\partial t}. \end{aligned}\right\} \tag{2.3}$$

Rewriting (2.1) in variables r and ψ, we have

$$\left.\begin{aligned} rn\frac{\partial u}{\partial r} + n^2 r^{\sigma+1}\frac{\partial u}{\partial \psi} + ru\frac{\partial n}{\partial r} + nu\sigma &= 0, \\ u\frac{\partial u}{\partial r} + n^{\gamma-2}\frac{\partial n}{\partial r} + n^{\gamma-1}r^\sigma\frac{\partial n}{\partial \psi} &= 0. \end{aligned}\right\} \tag{2.4a, b}$$

Because of the dearth of exact analytical solutions for flows with spherical and cylindrical symmetry, the attention will turn to approximate solutions. In expanding flows of this type, density is a decreasing function of radial distance for constant particle path function ψ. Consequently the density term in the momentum equation becomes negligible as $r \to \infty$, ψ fixed. This has been pointed out many times previously and has been termed the 'inertia dominated' region of the flow. If the terms involving density in (2.4 *b*) are neglected then $u = u(\psi)$ and, from the continuity equation, implies a density variation

$$nr^{\sigma+1} = \left(\frac{1}{u}\frac{du}{d\psi} + \frac{B(\psi)}{r}\right)^{-1}, \tag{2.5}$$

where $B(\psi)$ is an arbitrary function of integration. It can be seen that the expression for density reduces to the asymptotic steady state solution when $du/d\psi \to 0$, while for $du/d\psi$ and B of order unity

$$nr^{\sigma+1} \sim \frac{1}{u}\frac{du}{d\psi}, \tag{2.6}$$

for large r, and is mathematically equivalent to the expansion of a fixed mass of gas (Freeman & Grundy 1968; Grundy & Thomas 1969). However, when $(r/B)\,du/d\psi$ is of order unity, r large, we have a region of the flow field in which neither approximation is valid. As far as this paper is concerned, for flows with spherical symmetry, this is the region of interest for it is through this region that the steady state is approached.

The aim, therefore, is to construct a perturbation scheme based upon the solution (2.5) which is asymptotically valid for large r, ψ fixed. However, we must be prepared to incorporate in our expansion scheme the possibility of an approach to the steady state when $(r/B)\,du/d\psi$ is of order one. In view of this the following iteration scheme for the inviscid solution as $r \to \infty$ is proposed:

$$\left.\begin{aligned} nr^{\sigma+1} &= N_0(r,\psi)+r^{-\alpha}N_1(r,\psi)+\dots,\\ u &= u_0(\psi)+r^{-\alpha}u_1(r,\psi)+\dots. \end{aligned}\right\} \tag{2.7}$$

It will be assumed that these expansions exist throughout the region of interest. However, this will not necessarily be so near the inviscid zero density front where $n \to 0$. The expansions (2.7) are inserted into (2.4) giving

$$\left.\begin{aligned} &u_0 r\frac{\partial N_0}{\partial r}+N_0^2\frac{\partial u_0}{\partial \psi}-u_0N_0+r^{-\alpha}\left[N_0^2\frac{\partial u_1}{\partial \psi}+N_1 r\frac{\partial u_1}{\partial r}+2N_0N_1\frac{\partial u_0}{\partial \psi}\right.\\ &\qquad\left.+u_1 r\frac{\partial N_0}{\partial r}+u_0 r\frac{\partial N_1}{\partial r}-(\alpha+1)(N_0u_1+u_0N_1)\right]+O(r^{-2\alpha})=0,\\ &u_0\left\{r\frac{\partial u_1}{\partial r}-\alpha u_1\right\}+r^{\alpha-(\sigma+1)(\gamma-1)}\left\{N_0^{\gamma-2}r\frac{\partial N_0}{\partial r}-N_0^{\gamma-1}(\sigma+1)+N_0^{\gamma-1}\frac{\partial N_0}{\partial \psi}\right\}\\ &\qquad+O(r^{-\alpha})=0. \end{aligned}\right\} \tag{2.8a, b}$$

On examination of (2.8*b*) we conclude that

$$\alpha = (\sigma+1)(\gamma-1). \tag{2.9}$$

To zeroth order in $r^{-\alpha}$, (2.8*a*) is

$$u_0 r\frac{\partial N_0}{\partial r}+N_0^2\frac{\partial u_0}{\partial \psi}-u_0N_0=0,$$

giving

$$N_0=\left\{\frac{1}{u_0}\frac{du_0}{d\psi}+\frac{B(\psi)}{r}\right\}^{-1}, \tag{2.10}$$

where $B(\psi)$ is an arbitrary function of integration. Using the isentropic and perfect gas relations the corresponding terms for temperature and pressure are

$$T_0=\left\{\frac{1}{u_0}\frac{du_0}{d\psi}+\frac{B(\psi)}{r}\right\}^{1-\gamma}$$

and

$$p_0=\left\{\frac{1}{u_0}\frac{du_0}{d\psi}+\frac{B(\psi)}{r}\right\}^{-\gamma}, \tag{2.11a, b}$$

where the thermodynamic variables have been expanded as

$$\left.\begin{aligned} r^{(\sigma+1)(\gamma-1)}T &= T_0(r,\psi)+r^{-\alpha}T_1(r,\psi)+\dots,\\ r^{(\sigma+1)\gamma}p &= p_0(r,\psi)+r^{-\alpha}p_1(r,\psi)+\dots. \end{aligned}\right\} \tag{2.12}$$

Higher order terms in the asymptotic solutions can be obtained in a systematic manner. In particular, from (2.8*b*) using the above expression for N_0, we have

$$\begin{aligned} u_0\frac{\partial}{\partial r}\left(\frac{u_1}{r^{\alpha}}\right) &= r^{-(\alpha+1)}\left\{\frac{u_0'}{u_0}+\frac{B(\psi)}{r}\right\}^{-(\gamma+1)}\left[(\sigma+1)\left(\frac{u_0'}{u_0}\right)^2+\left(\frac{u_0'}{u_0}\right)'\right.\\ &\qquad\left.+\frac{1}{r}\left\{B'(\psi)+(2\sigma+1)B(\psi)\frac{u_0'}{u_0}\right\}+\frac{\sigma\{B(\psi)\}^2}{r^2}\right]. \end{aligned} \tag{2.13}$$

Integration of this expression in closed form is not possible except in a few special cases. Fortunately one of these is the case $\sigma = 2$, $\gamma = \frac{5}{3}$, i.e. the spherically symmetric flow of a monatomic gas. For these values of σ and γ (2.13) can be integrated to give

$$u_1+\left\{\frac{u_0'}{u_0}+\frac{B}{r}\right\}^{-\frac{5}{3}}\left[\frac{3B}{2ru_0}+\frac{3}{2}\frac{u_0'}{u_0^2}+\frac{3B'}{Bu_0}-L(\psi)\left\{\frac{u_0'}{u_0}+\frac{B}{r}\right\}^{\frac{5}{3}}r^2+\left\{\frac{3r}{2u_0B}+\frac{9u_0'r^2}{10u_0^2B^2}\right\}J_1(\psi)\right], \quad (2.14)$$

where

$$J_1(\psi)=\frac{6B'}{B}\frac{u_0'}{u_0}-\left(\frac{u_0'}{u_0}\right)',$$

the primes denoting differentiation with respect to ψ only, and $L(\psi)$ is an arbitrary function of integration. In the preliminary discussion concerning the motivation for these expansions three cases were mentioned. First the steady limit $du_0/d\psi \to 0$, secondly, $(r/B)\,du_0/d\psi = O(1)$, r large and thirdly $du_0/d\psi$ and B of order unity, r large. The general expansion scheme we have adopted includes all these cases; the r variation within the coefficients of $r^{-\alpha}$ has been retained in order to accommodate the second possibility. The arbitrary function $L(\psi)$ in the expression for u_1 is evaluated by examining the behaviour of u_1 in the third limit. In particular $L(\psi)$ is chosen so as to eliminate all the terms which become large in this case; this is achieved by the choice

$$L(\psi)=\left(\frac{u_0'}{u_0}\right)^{-\frac{2}{3}}\frac{9}{10u_0B^2}\left\{\frac{6B'u_0'}{Bu_0}-\left(\frac{u_0'}{u_0}\right)'\right\}. \quad (2.15)$$

Consequently the expression for u_1 can be written

$$u_1=-\left(\frac{r}{B}\right)^{\frac{2}{3}}\left(\frac{u_0B}{ru_0'}\right)^{\frac{5}{3}}\left[\frac{3}{2u_0}\left(1+\frac{Bu_0}{ru_0'}\right)^{-\frac{2}{3}}+\frac{3B'}{Bu_0'}\left(1+\frac{Bu_0}{ru_0'}\right)^{-\frac{5}{3}}+\frac{9u_0}{10}\left(\frac{ru_0'}{u_0B}\right)^2\left\{\frac{6B'}{Bu_0u_0'}-\frac{1}{u_0'^2}\left(\frac{u_0'}{u_0}\right)'\right\}\left\{\frac{5u_0B}{3ru_0'}+1-\left(1+\frac{Bu_0}{ru_0'}\right)^{\frac{5}{3}}\right\}\right]. \quad (2.16)$$

We now return to the case $(r/B)\,du_0/d\psi = O(1)$, r large. For the spherically symmetric expansion of a monatomic gas the expansion for u can be written

$$u=u_0(\psi)-\frac{1}{B^{\frac{2}{3}}r^{\frac{4}{3}}}\left(\frac{u_0B}{ru_0'}\right)^{\frac{5}{3}}\left[\frac{3}{2u_0}\left(1+\frac{Bu_0}{ru_0'}\right)^{-\frac{2}{3}}+\frac{3B'}{Bu_0'}\left(1+\frac{Bu_0}{ru_0'}\right)^{-\frac{5}{3}}+\frac{9u_0}{10}\left(\frac{ru_0'}{u_0B}\right)^2\left\{\frac{6B'}{Bu_0u_0'}-\frac{1}{u_0'^2}\left(\frac{u_0'}{u_0}\right)'\right\}\left\{\frac{5u_0B}{3ru_0'}+1-\left(1+\frac{Bu_0}{ru_0'}\right)^{\frac{5}{3}}\right\}\right]. \quad (2.17)$$

It is clear that $u(r,\psi)$ has been expanded for $rB^{\frac{1}{2}}$ large with $(r/B)\,u_0'$ fixed and of order unity. The remaining ψ dependence in the expansion is $O(1)$. In a similar manner it is possible to determine higher order terms in the expansions of density, temperature and pressure. However, this will not be done, the result for u is thought sufficient to indicate their behaviour.

The indeterminate nature of the asymptotic solution is quite apparent and, as observed above, is not unexpected. Both the arbitrary function $B(\psi)$ and the limiting gas velocity $u_0(\psi)$ are determined by the full inviscid solution. It is therefore unavoidable that the dependence on the initial conditions is lost. Nevertheless, although the exact role played by the initial conditions is not known, they can only enter through their dependence on the functions $B(\psi)$ and $u_0(\psi)$.

For the purposes of the present work several assumptions will be made regarding the structure of the flow field for large r. Perhaps the simplest picture will be that representing the particle path lines, which, to this order, coincide with the characteristic directions. The variation of u_0 and B are taken so that the resulting particle line structure is as shown in figure 1, where the origin has been

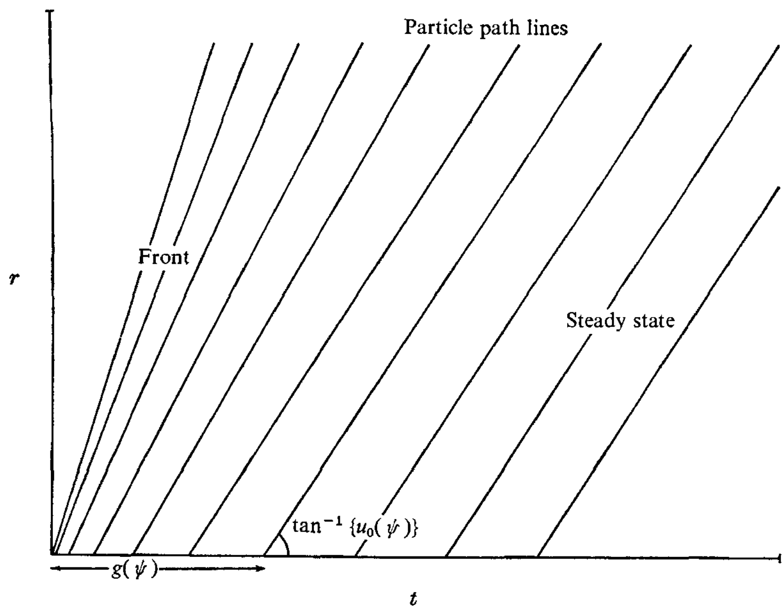

FIGURE 1. Schematic representation of far field particle path line structure.

chosen so that the figure represents the flow for large r. By virtue of the fact that $u_0 = u_0(\psi)$, the particle path lines must be straight. In addition it is assumed that u_0 decreases as we move away from the front for constant r until the steady state is reached, where the particle path lines become parallel. The limit $B \to 0$ corresponds to the purely unsteady flow of a fixed mass of gas and consequently it is apparent that this occurs as the front of the expansion is approached, due to the fact that this part of the flow field will be largely unaffected by the remainder of the gas. On the other hand as the steady state region is approached $u_0 \to$ constant and the function $B(\psi)$ will also become independent of the particle path line.

Bearing in mind these considerations several quantitative remarks can be made about the far field flow structure. The particle path lines can be represented in parametric form for large r by the system

$$r = u_0(\psi)\{t - g(\psi)\} = u_0\{t - g(u_0)\}. \tag{2.18}$$

For each ψ this defines a straight line with slope $u_0(\psi)$ and an intercept $g(\psi)$ on the t axis. Differentiating this with respect to r at constant t we have

$$\left(\frac{\partial u_0}{\partial r}\right)_t = \left\{t - g - u_0 \frac{dg}{d\psi}\right\}^{-1}.$$

For r large and $(r/B)\,du_0/d\psi = O(1)$,

$$\left(\frac{\partial\psi}{\partial r}\right)_t = nr^{\sigma} \sim \left\{\frac{r}{u_0}\frac{du_0}{d\psi} + B(\psi)\right\}^{-1}.$$

Now $\psi = \psi(u_0)$ to this degree of approximation and hence

$$\frac{d\psi}{du_0}\left(\frac{\partial u_0}{\partial r}\right)_t \sim \left\{\frac{r}{u_0}\frac{du_0}{d\psi} + B(\psi)\right\}^{-1} = \left\{[t - g(\psi)]\frac{du_0}{d\psi} + B(\psi)\right\}^{-1}.$$

Equating the two expressions for $(\partial u_0/\partial r)_t$ we have

$$B(\psi) = -u_0\frac{dg}{d\psi}. \tag{2.19}$$

These expressions will be utilized when the example of § 4 is considered.

3. The Boltzmann equation for spherical flow

3.1. *The near equilibrium solution of the Boltzmann equation*

The realization that the inviscid solution is an asymptotic approximation to the full solution of the Boltzmann equation leads at once to the question as to whether approximation is uniform. It is with this question that this section is concerned.

Boltzmann's equation for unsteady flow with spherical symmetry can be written

$$\frac{\partial f}{\partial t} + (\xi_1 + u)\left\{\frac{\partial f}{\partial r} - \frac{\partial f}{\partial \xi_1}\frac{\partial u}{\partial r}\right\} - \frac{\partial f}{\partial \xi_1}\frac{\partial u}{\partial t} + \frac{\Gamma^2}{r}\frac{\partial f}{\partial \xi_1} - \frac{\Gamma(\xi_1 + u)}{r}\frac{\partial f}{\partial \Gamma} = ANI. \tag{3.1}$$

The right-hand side expresses the change in the distribution function due to collisions and can be written (Chapman & Cowling 1960) in the usual notations

$$NI_s = \int \cdots \int (f_1' f_2' - f_1 f_2)\, d\xi_2\, d\eta_2\, d\zeta_2\, k_{12}\, d\mathbf{k}. \tag{3.2}$$

The variables in (3.1) have been non-dimensionalized as in § 2, and in addition the non-dimensional peculiar molecular velocities

$$\xi_1 = \frac{\xi' - u'}{a_0'}, \quad \Gamma = \left\{\frac{\eta'^2 + \zeta'^2}{a_0'^2}\right\}^{\frac{1}{2}}, \tag{3.3}$$

are respectively in the radial and transverse directions. η' and ζ' are the molecular velocities in the θ and ϕ directions. The parameter A is proportional to the inverse source Knudsen number and is assumed large, compatible with near continuum conditions at $r = L$, $t = 0$. Re-written in the particle path variables, (3.1) becomes

$$u\frac{\partial f}{\partial r} + \xi_1\frac{\partial f}{\partial r} + \xi_1 N r^2\frac{\partial f}{\partial \psi} - \xi_1\frac{\partial f}{\partial \xi_1}\frac{\partial u}{\partial r} - \xi_1 N r^2\frac{\partial f}{\partial \xi_1}\frac{\partial u}{\partial \psi}$$
$$- u\frac{\partial f}{\partial \xi_1}\frac{\partial u}{\partial r} + \frac{\Gamma^2}{r}\frac{\partial f}{\partial \xi_1} - \frac{\Gamma(\xi_1 + u)}{r}\frac{\partial f}{\partial \Gamma} = ANI, \tag{3.4}$$

where the density N is defined by

$$N = 2\pi \int_0^\infty \int_{-\infty}^\infty \Gamma f \, d\xi_1 \, d\Gamma, \tag{3.5}$$

and u is the gas velocity.

The Chapman–Enskog solution to the Boltzmann equation can be regarded as a near equilibrium expansion in powers of A^{-1} for the distribution function f. This can be written for the present problem,

$$f = f_0 \left\{ 1 - A^{-1} \left[\left(\frac{2k}{m} \right)^{\frac{1}{2}} \frac{1}{nT^{\frac{1}{2}}} A_r(\mathscr{C}) \left\{ \frac{\partial T}{\partial r} + Nr^2 \frac{\partial T}{\partial \psi} \right\} - \frac{B_1(\mathscr{C})}{n} \frac{\partial \mathbf{u}}{\partial \mathbf{r}} \right] + O(A^{-2}) \right\}, \tag{3.6}$$

where $\mathscr{C} = (\xi_1^2 + \Gamma^2)/2T$. The function f_0 is the Maxwellian distribution given by

$$f_0 = \frac{n}{(2\pi T)^{\frac{3}{2}}} \exp - \left\{ \frac{\xi_1^2 + \Gamma^2}{2T} \right\}. \tag{3.7}$$

The function $A_r(\mathscr{C})$, the tensor $B_1(\mathscr{C})$ and the vector–tensor product notation are defined in Chapman & Cowling (1960). The thermodynamic variables are now expanded as

$$\left. \begin{aligned} T &= \tau_0 + A^{-1}\tau_1 + \dots, \\ N &= R_0 + A^{-1}R_1 + \dots, \\ u &= V_0 + A^{-1}V_1 + \dots, \\ \psi &= \psi_0 + A^{-1}\psi_1 + \dots. \end{aligned} \right\} \tag{3.8}$$

τ_0, R_0, V_0 and ψ_0 are just the inviscid thermodynamic variables and particle path function respectively. The asymptotic behaviour of the near equilibrium expansion can be investigated using the inviscid solution developed in the previous section. By substituting the expansions (3.8) into the Chapman–Enskog solution the behaviour of the second term in equation (3.6) can be examined for large r. The assumption of Maxwell molecules enables this second term to be explicitly evaluated. This procedure gives

$$\begin{aligned} f = f_0 \Bigg\{ 1 + A^{-1} \Bigg[\left(\frac{2k}{m} \right)^{\frac{1}{2}} a_1 \mathscr{C}_r \Bigg\{ B^{\frac{1}{2}} (rB^{\frac{1}{2}})^{\frac{1}{3}} \Bigg[\left(\frac{u_0' r}{u_0 B} + 1 \right) \left(\frac{u_0' r}{u_0 B} + \frac{4}{3} \right) + \frac{2}{3} \left(\frac{r u_0'}{u_0 B} \right)^2 \left(\frac{u_0}{u_0'} \right)^2 \left(\frac{u_0'}{u_0} \right)' \\ + \frac{2}{3} \frac{B' u_0}{B u_0'} \left(\frac{r u_0'}{u_0 B} \right) \Bigg] \Bigg/ \left(\frac{u_0' r}{u_0 B} + 1 \right)^{\frac{4}{3}} \\ + b_1 B r \left\{ \frac{2\xi_1^2 - \Gamma^2}{2\tau_0} \right\} + O\{(rB^{\frac{1}{2}})^{-\frac{4}{3}}\} \Bigg] + O(A^{-2}) \Bigg\}, \end{aligned} \tag{3.9}$$

where a_1 and b_1 are constants evaluated in Chapman & Cowling (1960). $B(\psi_0)$ is the function defined in § 2, $\mathscr{C}_r = \xi_1/(2\tau_0)^{\frac{1}{2}}$ and primes denote differentiation with respect to ψ_0. For the scaled velocities $\mathscr{C}_r$, $\mathscr{C}$, etc., remaining fixed, it is apparent that the near equilibrium expansion will break down when

$$r = O(A/B(\psi_0)). \tag{3.10}$$

The location of the region of non-uniformity therefore depends on the parameter A and also on the function $B(\psi_0)$. Returning briefly to the inviscid solution we define a new variable

$$r_1 = \frac{rB(\psi_0)}{A}. \tag{3.11}$$

In terms of this new variable (2.10) and (2.11) become for $\sigma = 2$ and $\gamma = \frac{5}{3}$,

$$\left.\begin{aligned} n_0' &= \frac{N_0 A^2}{B} = r_1^{-2}\left\{\frac{u_0' A r_1}{u_0 B^2}+1\right\}^{-1}\{1+O(B^{\frac{2}{3}}A^{-\frac{4}{3}})\}, \\ T_0' &= \frac{T_0 A^{\frac{4}{3}}}{B^{\frac{2}{3}}} = r_1^{-\frac{4}{3}}\left\{\frac{u_0' A r_1}{u_0 B^2}+1\right\}^{-\frac{2}{3}}\{1+O(B^{\frac{2}{3}}A^{-\frac{4}{3}})\}, \\ p_0' &= \frac{p_0 A^{\frac{10}{3}}}{B^{\frac{5}{3}}} = r_1^{-\frac{10}{3}}\left\{\frac{u_0' A r_1}{u_0 B^2}+1\right\}^{-\frac{5}{3}}\{1+O(B^{\frac{2}{3}}A^{-\frac{4}{3}})\}. \end{aligned}\right\} \tag{3.12}$$

As noted earlier in the paper (§ 2) the region of interest is now confined to $(u_0' A/B^2) = O(1)$, i.e. close to the steady flow limit. The case of u_0'/B^2 of order one will be discussed later in § 3.4.

3.2. *The outer solution*

It has been shown above that the inviscid solution breaks down when

$$r = O(A/B(\psi_0)),$$

and consequently equations (3.11) give the order of magnitude of the thermodynamic variables in this region. New outer variables are defined as

$$r_1 = \frac{rB}{A}, \quad n_1 = \frac{NA^2}{B}, \tag{3.13}$$

together with the molecular velocities

$$\phi = \xi_1 (A^2/B)^{\frac{1}{3}} \quad \text{and} \quad \chi = \Gamma (A^2/B)^{\frac{1}{3}}.$$

Inserting this scaling into the Boltzmann equation (3.4) we have

$$\begin{aligned} -u\frac{\partial f}{\partial \phi}\frac{\partial u}{\partial r_1}\left(\frac{A^2}{B}\right)^{\frac{1}{3}} + u\frac{\partial f}{\partial r_1} - \phi\frac{\partial f}{\partial \phi}\frac{\partial u}{\partial r_1} - \frac{\chi u}{r_1}\frac{\partial f}{\partial \chi} - \phi n_1 r_1^2 \frac{\partial f}{\partial \phi}\frac{\partial u}{\partial \psi}\frac{A}{B^2} \\ + \left(\frac{A^2}{B}\right)^{-\frac{1}{3}}\left\{\phi\frac{\partial f}{\partial r_1} + \frac{\chi^2}{r_1}\frac{\partial f}{\partial \phi} - \frac{\chi\phi}{r_1}\frac{\partial f}{\partial \chi} + \phi n_1 r_1^2 \frac{\partial f}{\partial \psi}\frac{A}{B^2}\right\} = n_1 I, \end{aligned} \tag{3.14}$$

where the terms involving A multiplied by a ψ derivative are retained at each approximation level in order to bring out the essential features of the flow when $u_0' A/B^2 = O(1)$. Taking the zeroth-order moment of (3.14) we have

$$u\frac{\partial n_1}{\partial r_1} + n_1\frac{\partial u}{\partial r_1} + n_1^2 r_1^2 \frac{\partial u}{\partial \psi}\frac{A}{B^2} + \frac{2un_1}{r_1} = 0,$$

and the first ϕ moment is

$$n_1 u\frac{\partial u}{\partial r_1} + \left(\frac{B}{A^2}\right)^{\frac{2}{3}}\left\{\frac{\partial \overline{\phi^2}}{\partial r_1} + n_1 r_1^2 \frac{\partial \overline{\phi^2}}{\partial \psi}\frac{A}{B^2} - \frac{\overline{\chi^2}}{r_1} + \frac{2\overline{\phi^2}}{r_1}\right\} = 0, \tag{3.15}$$

where $\quad \overline{\phi^2} = 2\pi \int_0^\infty \int_{-\infty}^\infty \phi^2 \chi f \, d\phi \, d\chi \quad$ and $\quad \overline{\chi^2} = 2\pi \int_0^\infty \int_{-\infty}^\infty f\chi^3 \, d\phi \, d\chi.$

The following expansions are made in the outer region

$$\left.\begin{aligned} u &= U_0(\Psi_0) + U_1 (B/A^2)^{\frac{2}{3}} + \ldots, \\ n_1 &= \rho_0(\Psi_0, r_1) + \rho_1 (B/A^2)^{\frac{2}{3}} + \ldots, \\ \psi &= \Psi_0 + \Psi_1 (B/A^2)^{\frac{2}{3}} + \ldots, \end{aligned}\right\}$$

and

$$f = F_0 + F_1 (B/A^2)^{\frac{1}{3}} + \ldots. \tag{3.16}$$

Equations (3.15) now give, neglecting terms $O\{(B/A^2)^{\frac{2}{3}}\}$,

$$U_0 \frac{\partial \rho_0}{\partial r_1} + \rho_0^2 r_1^2 \left[\frac{A}{B^2} \frac{\partial U_0}{\partial \Psi_0}\right] + \frac{2U_0\rho_0}{r_1} = 0. \tag{3.17}$$

Matching with the inner solution for fixed Ψ_0 gives,

$$\rho_0 r_1^2 = \left\{\left[\frac{U_0' A}{U_0 B_2}\right] r_1 + 1\right\}^{-1}, \tag{3.18}$$

$$U_0 = u_0$$

and

$$\Psi_0 = \psi_0.$$

The expansions for u and n are introduced into (3.14) and a new variable

$$s_1 = r_1 \left[\frac{U_0' A}{U_0 B^2}\right] \tag{3.19}$$

is defined. Neglecting terms $O\{(B/A^2)^{\frac{1}{3}}\}$,

$$(s_1 + 1) \frac{\partial F_0}{\partial s_1} - \chi \frac{\partial F_0}{\partial \chi} \frac{(1+s_1)}{s_1} - \phi \frac{\partial F_0}{\partial \phi} = \frac{J(\Psi_0)}{s_1^2} I, \tag{3.20}$$

where $J(\Psi_0) = U_0' A / U_0 B^2$ and I_0 is the zeroth-order term in the expansion of the collision integral. Changing to the molecular velocity variables

$$\chi_1 = \chi s_1, \quad \phi_1 = \phi(1+s_1), \tag{3.21}$$

and putting $y_1 = (1+s_1)/s_1$ we finally have

$$\frac{y_1}{y_1 - 1} \frac{\partial F_0}{\partial y_1} + J(\Psi_0) I_0 = 0. \tag{3.22}$$

The form of the collision integral, and in particular I_0, has so far remained unspecified. In order to make any progress at obtaining a solution of (3.22) by present methods I_0 must be known, at least sufficiently so that its moments must be expressible in terms of moments of the distribution function. In previous work (Freeman 1967; Freeman & Grundy 1968; Freeman & Thomas 1969) it was found that, for the Maxwell molecule collision model, the collision integral simplified in such a way that integrals of I_0 over velocity space could be expressed in terms of moments of F_0. Furthermore, it was also shown that, up to second-order moments, a more simplified collision model, namely the BGK model, would yield the same set of moment equations as the full Boltzmann equation for Maxwell molecules. As far as the present work is concerned the BGK model will be used to compute moments of the distribution function, the inference being that the moments obtained will be identical to those which would have been found using the full Maxwell molecule collision integral.

For the BGK model

$$I_0 = F^* - F_0, \tag{3.23}$$

where F^* is the local Maxwellian distribution given by

$$F^* = \frac{\rho_0}{(2\pi T_0)^{\frac{3}{2}}} \exp - \left\{\frac{\phi^2 + \chi^2}{2T_0}\right\}, \tag{3.24}$$

and T_0 is the local thermodynamic temperature given by

$$T_0 = 2\pi \int_0^\infty \int_{-\infty}^\infty (\phi^2 + \chi^2)\, \chi F_0 \, d\phi \, d\chi. \tag{3.25}$$

Substituting for ρ_0 and making the change of molecular velocity variable in (3.24), (3.22) becomes

$$\frac{y_1}{y_1 - 1} \frac{\partial F_0}{\partial y_1} - J(\Psi_0) F_0 + \frac{G(\Psi_0)(y_1 - 1)^3}{(2\pi T_0)^{\frac{3}{2}} y_1} \exp - \left\{ \frac{\dfrac{\phi_1^2 (y_1 - 1)^2}{y_1^2} + \chi_1^2 (y_1 - 1)^2}{2T_0} \right\} = 0, \tag{3.26}$$

where
$$G(\Psi_0) = \left(\frac{U_0' A}{U_0^{\frac{4}{3}} B^2} \right)^3. \tag{3.27}$$

The solution of (3.26) for F_0 can be written

$$F_0 = -G(\Psi_0)\, y_1^J e^{J y_1} \int_\infty^{y_1} \frac{(y_2 - 1)^4}{(2\pi T_0)^{\frac{3}{2}}} y_2^{J-2} e^{-J y_2} \exp - \left\{ \frac{\dfrac{\phi_1^2 (y_2 - 1)^2}{y_2^2} + \chi_1^2 (y_2 - 1)^2}{2T_0} \right\} dy_2, \tag{3.28}$$

where $J \equiv J(\Psi_0)$. It should be stressed at this point in the analysis that Ψ_0 plays the role of a parameter as far as the outer solution is concerned, considerably simplifying the flow structure.

We now define the second-order moments

$$\left. \begin{aligned} \overline{\phi_1^2} &= 2\pi \int_{-\infty}^\infty \int_0^\infty \phi_1^2 F_0 \chi_1 \, d\chi_1 \, d\phi_1, \\ \overline{\chi_1^2} &= 2\pi \int_{-\infty}^\infty \int_0^\infty \chi_1^3 F_0 \, d\chi_1 \, d\phi_1, \end{aligned} \right\} \tag{3.29}$$

and substituting for F_0 from (3.28), integrating over velocity space (ϕ_1, χ_1), gives

$$\left. \begin{aligned} \overline{\phi_1^2} &= -\frac{G(\Psi_0)\, e^{J y_1}}{y_1^J} \int_\infty^{y_1} \frac{T_0 y_2^{J+1} e^{-J y_2}}{y_2 - 1} dy_2 \\ \text{and} \quad \overline{\chi_1^2} &= -\frac{2G(\Psi_0)\, e^{J y_1}}{y_1^J} \int_\infty^{y_1} \frac{T_0 y_2^{J-1} e^{-J y_2}}{y_2 - 1} dy_2. \end{aligned} \right\} \tag{3.30}$$

The relationship between these moments and the temperature can be found from the identity relating the diagonal elements of the stress tensor to density and temperature, namely

$$\frac{(y_1 - 1)^2}{y_1^2} \overline{\phi_1^2} + (y_1 - 1)^2 \overline{\chi_1^2} = 3K(\Psi_0) T_0, \tag{3.31}$$

where
$$K(\Psi_0) = \left(\frac{U_0' A}{U_0 B^2} \right)^2.$$

Substituting for $\overline{\chi_1^2}$ and $\overline{\phi_1^2}$ from (3.30) into (3.31) an integral equation for T_0 can be obtained. The integrals which appear can be eliminated between the equations at our disposal to give a second-order differential equation for the temperature T_0. This equation is reducible to the confluent hypergeometric equation

$$\left.\begin{aligned} & w_1 \frac{\partial^2 Y_0}{\partial w_1^2} + (2\beta + \tfrac{5}{3} + J - w_1)\frac{\partial Y_0}{\partial w_1} - \beta Y_0 = 0, \\ \text{where} \quad & T_0 = \{Jy_1\}^{\beta}(y_1 - 1)^2 y_1^{-\frac{2}{3}} Y_0, \\ & w_1 = Jy_1 \\ \text{and} \quad & 2\beta = -(\tfrac{2}{3} + J) \pm \sqrt{(\tfrac{4}{3}J + J^2 + 4)}. \end{aligned}\right\} \tag{3.32}$$

(In the equation defining β the negative sign will be taken.) The final solution for T_0 can be written

$$T_0 = D(\Psi_0)(Jy_1)^{\beta}(y_1 - 1)^2 y_1^{-\frac{2}{3}} \Psi_*(\beta, 2\beta + \tfrac{5}{3} + J; Jy_1), \tag{3.33}$$

where the Ψ_* function is the confluent hypergeometric function and $D(\Psi_0)$ is a constant of integration. The second confluent hypergeometric function, which is also a solution of (3.32), has been dismissed by virtue of its behaviour as $y_1 \to \infty$, for fixed Ψ_0, i.e. the inviscid limit. Now

$$Jy_1 = (U_0 r_1)^{-1} + J$$

and thus the inviscid limit, $r_1 \to 0$, Ψ_0 fixed, corresponds to $Jy_1 \to \infty$. Taking this limit in (3.33)

$$T_0 \sim D(\Psi_0)(y_1 - 1)^2 y_1^{-\frac{2}{3}}\{1 + O(r_1) + \ldots\}.$$

Matching with the inner solution, which can be written

$$T' \sim (U_0 J)^{\frac{4}{3}}(y_1 - 1)^2 y_1^{-\frac{2}{3}}\{1 + O\{(B/A^2)^{\frac{2}{3}}\}\},$$

the function $D(\Psi_0)$ is determined as

$$D(\Psi_0) = (U_0 J)^{\frac{4}{3}}. \tag{3.34}$$

Thus the solution for T_0 which matches with the inviscid solution, becomes

$$T_0 = U_0^{\frac{4}{3}} J^{\frac{4}{3}+\beta} y_1^{\beta-\frac{2}{3}}(y_1 - 1)^2 \Psi_*(\beta, 2\beta + \tfrac{5}{3} + J; Jy_1). \tag{3.35}$$

The integrals occurring in equations (3.30) can be found in terms of T_0 and its first derivatives with respect to y_1, hence the second-order moments $\overline{\phi_1^2}$ and $\overline{\chi_1^2}$ can be determined. These are

$$\left.\begin{aligned} \overline{\phi_1^2} &= -U_0^{\frac{10}{3}} \tfrac{3}{2} J^{\beta+\frac{10}{3}} y_1^{\beta+\frac{4}{3}}\{(\beta - \tfrac{2}{3})\Psi_*(\beta, 2\beta + \tfrac{5}{3} + J; Jy_1) \\ &\qquad - \beta y_1 \Psi_*(\beta + 1, 2\beta + \tfrac{8}{3} + J; Jy_1)\} \\ \text{and} \quad \overline{\chi_1^2} &= -U_0^{\frac{10}{3}} \tfrac{3}{2} J^{\beta+\frac{10}{3}} y_1^{\beta-\frac{2}{3}}\{\beta y_1 \Psi_*(\beta + 1, 2\beta + \tfrac{8}{3} + J; Jy_1) \\ &\qquad - (\beta + \tfrac{4}{3})\Psi_*(\beta, 2\beta + \tfrac{5}{3} + J; Jy_1)\}. \end{aligned}\right\} \tag{3.36}$$

The behaviour of the solution for $r_1 \to \infty$, Ψ_0 fixed, can be found from the above solution. In particular for temperature, taking the limit $r_1 \to \infty$ and noting that

$$Jy_1 = J + \frac{1}{(U_0 r_1)}, \quad y_1 = 1 + \left[\frac{U_0 B^2}{U_0' A}\right] r_1^{-1},$$

then

$$T_0 r_1^2 = U_0^{-\frac{2}{3}} J^{\beta-\frac{2}{3}} \Psi_*(\beta, 2\beta + \tfrac{5}{3} + J; J) + O(1/r_1). \tag{3.37}$$

Thus for each Ψ_0, T_0 behaves like r_1^{-2} as $r_1 \to \infty$ provided of course that $J\,(= U_0'A/U_0^2B^2)$ does not become zero. In the case of the steady flow limit a different approach is necessary.

3.3. *The steady flow limit*

It was observed in § 2 that the steady flow limit of the asymptotic inviscid solution was associated with u_0' approaching zero. In addition it was seen that the function $B(\Psi)$ became constant. It will therefore be instructive to examine the outer solution for the second-order moments and the Boltzmann equation itself in this limit. It will be recalled that the approximation procedure followed in the outer region was intended to retain the essential behaviour of the solution as the steady limit was approached. The terms involving a ψ derivative and the factor A were therefore retained. As noted previously

$$Jy_1 = \frac{U_0'A}{U^2B^2} + \frac{1}{U_0 r_1} = J + \frac{1}{U_0 r_1},$$

thus for fixed r_1 and $J \to 0$, $Jy_1 \sim r_1^{-1}$. From the second of equations (3.32) β approaches $-\frac{4}{3}$ as $J \to 0$, and so

$$T_0 = U_0^{\frac{4}{3}}\Psi_*(-\tfrac{4}{3}, -1;\, 1/U_0 r_1) + O(J), \tag{3.38}$$

as the steady state is approached.

The remaining second-order moments must be treated with more caution. First, as $J \to 0$, the scaled molecular velocities χ_1 and ϕ_1 must be redefined. Now,

$$\chi_1 = \chi s_1 = \chi J r_1 U_0$$

and

$$\phi_1 = \phi + O(J).$$

Thus we define

$$\left.\begin{aligned} \chi_2 &= \chi_1/JU_0 \\ \phi_2 &= \phi, \end{aligned}\right\} \tag{3.39}$$

and as the appropriate molecular velocities, noting that the velocity ϕ_2 is not now scaled with any spatial co-ordinate. Furthermore, it is apparent that the second-order moment must be redefined as

$$\left.\begin{aligned} \overline{\chi_2^2} &= 2\pi\int_{-\infty}^{\infty}\int_0^{\infty} \chi_2^3 F_0\, d\chi_2\, d\phi_2 = (U_0 J)^{-4}\,\overline{\chi_1^2} \\ \overline{\phi_2^2} &= 2\pi\int_{-\infty}^{\infty}\int_0^{\infty} \phi_2^2 F_0 \chi_2\, d\chi_2\, d\phi_2 = (U_0 J)^{-2}\,\overline{\phi_1^2}. \end{aligned}\right\} \tag{3.40}$$

and

We conclude that $\overline{\chi_2^2}$ and $\overline{\phi_2^2}$ are the second-order moments for the steady spherically symmetric flow of a monatomic gas into vacuo for Maxwell molecules. To see this we return to equation (3.36) and take the limit $J \to 0$, then

$$\left.\begin{aligned} \overline{\phi_2^2} &= (U_0 J)^{-2}\,\overline{\phi_1^2} = U_0^{\frac{4}{3}}\{3\Psi_*(-\tfrac{4}{3}, -1;\, 1/U_0 r_1) - (2/U_0 r_1)\,\Psi_*(-\tfrac{1}{3}, 0;\, 1/U_0 r_1)\}, \\ \overline{\chi_2^2} &= (U_0 J)^{-4}\,\overline{\chi_1^2} = 3U_0^{\frac{4}{3}} r_1^2\{\Psi_*(-\tfrac{4}{3}, -1;\, 1/U_0 r_1) + (2/3U_0 r_1)\,\Psi_*(-\tfrac{1}{3}, 0;\, 1/U_0 r_1)\}. \end{aligned}\right\} \tag{3.41}$$

These expressions for the second-order moments, including temperature, are equivalent to those obtained using the steady flow model (Freeman 1967; Edwards & Cheng 1966; Hamel & Willis 1966).

The Boltzmann equation itself can be shown to reduce to its steady counterpart in the limit $J \to 0$. Equation (3.22) becomes in the new velocity space χ_2, ϕ_2

$$r_1 \frac{\partial F_0}{\partial r_1} = \frac{1}{U_0 r_1} I_0(r_1, \chi_2, \phi_2). \tag{3.42}$$

3.4. *The case* U_0'/B^2 *of* $O(1)$

In the previous section the approach to the steady-state solution for the spherically symmetric flow has been discussed and explicit solutions for temperature and the other second-order moments have been obtained. In that solution we make the assumption $U_0'A/B^2 = O(1)$, which marks out the transition region spanning the unsteady and steady states. In this section of the paper some brief comments are made concerning the region of the flow where U_0'/B^2 is order unity. In order to do this the results of an earlier paper (Grundy & Thomas 1969) will be recalled which dealt with the spherically symmetric expansion of a fixed mass of gas into vacuum, which in the present context corresponds to $B(\Psi_0) \to 0$, U_0' of $O(1)$.

For $B(\psi) \to 0$, the resulting particle path structure for large r is that of a centred wave with $u_0(\psi)$ varying for different lines. For this type of flow with spherical symmetry,

$$\left.\begin{aligned} u_0 &\sim u_0(\psi), \\ nr^3 &\sim u_0 \Big/ \frac{du_0}{d\psi} \\ \text{and} \qquad Tr^2 &\sim \left(u_0 \Big/ \frac{du_0}{d\psi}\right)^{\frac{2}{3}}. \end{aligned}\right\} \tag{3.43}$$

It was shown by Grundy & Thomas that with this density variation the Maxwellian distribution function was a uniform approximation to the solution of the Boltzmann equation for large r, ψ fixed. However, the near equilibrium expansion is not uniformly valid but has to be replaced by a different outer expansion, valid when $r = O(A^{\frac{1}{2}})$, $\psi \equiv \psi(r/t) = O(1)$, but in both the outer and inner expansions for the distribution function the Maxwellian distribution was the leading term. If in the present problem U_0'/B^2 is order unity, then from (2.10) and (2.11) the inviscid thermodynamic variables can be written for large r as

$$\left.\begin{aligned} u &= u_0(\psi) + O(r^{-2}), \\ nr^3 &= u_0 \Big/ \frac{du_0}{d\psi} + O(r^{-1}), \\ Tr^2 &= \left(u_0 \Big/ \frac{du_0}{d\psi}\right)^{\frac{2}{3}} + O(r^{-1}). \end{aligned}\right\} \tag{3.44}$$

These solutions have the same mathematical structure as relations (3.43) for the expansion of a fixed mass of gas, and it is conjectured that in this case a similar non-equilibrium analysis would apply but with a considerable complexity arising in the higher order terms. This would mean that when $r = O(A^{\frac{1}{2}})$ and ψ fixed such that u_0'/B^2 is $O(1)$, the leading term of an asymptotic approximation to the solution of the Boltzmann equation would be the Maxwellian distribution given by

$$F_0 = (2\pi)^{-\frac{3}{2}} \exp - \left\{ \frac{(\xi_{12}^2 + T_2^2)\, r_2^2}{2\left(u_0 \Big/ \frac{du_0}{d\psi}\right)^{\frac{2}{3}}} \right\}, \tag{3.45}$$

where ξ_{12} is the peculiar molecular velocity relative to the mean gas velocity $u_0(\psi)$ and r_2, ξ_{12} and Γ_2 are order one variables in the outer region $r = O(A^{\frac{1}{3}})$, ψ fixed.

Taking the limit $U_0' A/B^2 \to \infty$ in (3.35) for T_0 gives

$$T_0 \sim \left(\frac{U_0 B^2}{U_0' A}\right)^{\frac{2}{3}} r_1^{-2}, \tag{3.46}$$

and provides matching with the solution for temperature in the region

$$U_0'/B^2 = O(1).$$

4. Illustration of theory

In this section of the paper specific variations of u_0 and B will be postulated and the resulting flow field will be computed. The choice of u_0 and B must comply with the qualitative and quantitative observations of § 2. Treating r and Ψ_0 as the independent variables we put $\Psi^* = T - \Psi_0$ where $T \to \infty$, T being the total mass of gas available for the expansion. Thus $\Psi^* \to 0$ at the front of the expansion and $\Psi^* \to \infty$ as the steady state is approached. We take

$$u_0 = \frac{k_1(1+k_2+\Psi^*)}{1+\Psi^*} \quad \text{giving} \quad \frac{du_0}{d\Psi_0} = -\frac{du_0}{d\Psi^*} = \frac{k_1 k_2}{(1+\Psi^*)^2},$$

and

$$B = \frac{\alpha k_1(1+k_2+\Psi^*)\,\Psi^*}{(1+\Psi^*)^2}. \tag{4.1}$$

The first choice has been made so that as $\Psi^* \to 0$, $u_0 \to k_1(1+k_2)$ and as $\Psi^* \to \infty$, $u_0 \to k_1$. The behaviour of u_0 as $\Psi^* \to 0$ is highly artificial as it is not expected that our expansion procedure is valid there. However, the example constructed will serve our primary purpose of illustrating the approach to the steady state through the transition region. The value of B is chosen so that as $\Psi^* \to 0$, $B \to 0$ and as $\Psi^* \to \infty$, $B \to$ constant.

The solution developed in § 3 applied when $u_0'/B^2 = U_0'/B^2 = O(A^{-1})$. In this example this implies that $\Psi^* = O(A^{\frac{1}{2}}/\alpha)$, and, writing $\Psi_1^* = (\alpha/A^{\frac{1}{2}})\Psi^*$ it is now assumed that Ψ_1^* is of order one. In this limit we have

$$u_0 = k_1 + O(\alpha A^{-\frac{1}{2}}), \quad \frac{du_0}{d\Psi_0} = \frac{k_1 k_2}{(A/\alpha^2)\,\Psi_1^{*2}} + O(\alpha^3 A^{-\frac{3}{2}}) \tag{4.2}$$

and

$$B = \alpha k_1 + O(\alpha A^{-\frac{1}{2}}).$$

From (2.19)

$$\frac{dg}{d\Psi^*} = \frac{\alpha \Psi^*}{1+\Psi^*},$$

and hence

$$g = A^{\frac{1}{2}}\Psi_1^* + O(1).$$

Thus the particle path lines are given by

$$r_1 = k_1 t \frac{\alpha}{A} - k_1 \Psi_1^* \frac{\alpha}{A^{\frac{1}{2}}},$$

i.e.

$$\Psi_1^* = \frac{A^{\frac{1}{2}}}{\alpha k_1}\left\{k_1 t \frac{\alpha}{A} - r_1\right\}. \tag{4.3}$$

The parameters occurring in the outer solution can now be found to a zeroth-order approximation in $\alpha A^{-\frac{1}{2}}$, these are given by

$$J(\Psi_1^*) = \frac{k_2}{k_1^3 \Psi_1^{*2}},$$

$$\beta = -\frac{1}{2k_1^3 \Psi_1^{*2}}\left\{\frac{2k_1^3 \Psi_1^{*2}}{3} + k_2 + \sqrt{\left(\tfrac{4}{3} k_2 k_1^3 \Psi_1^{*2} + k_2^2 + 4 k_1^6 \Psi_1^{*4}\right)}\right\}, \tag{4.4}$$

and in addition,

$$y_1 = 1 + \frac{k_1^2 \Psi_1^{*2}}{k_2 r_1}.$$

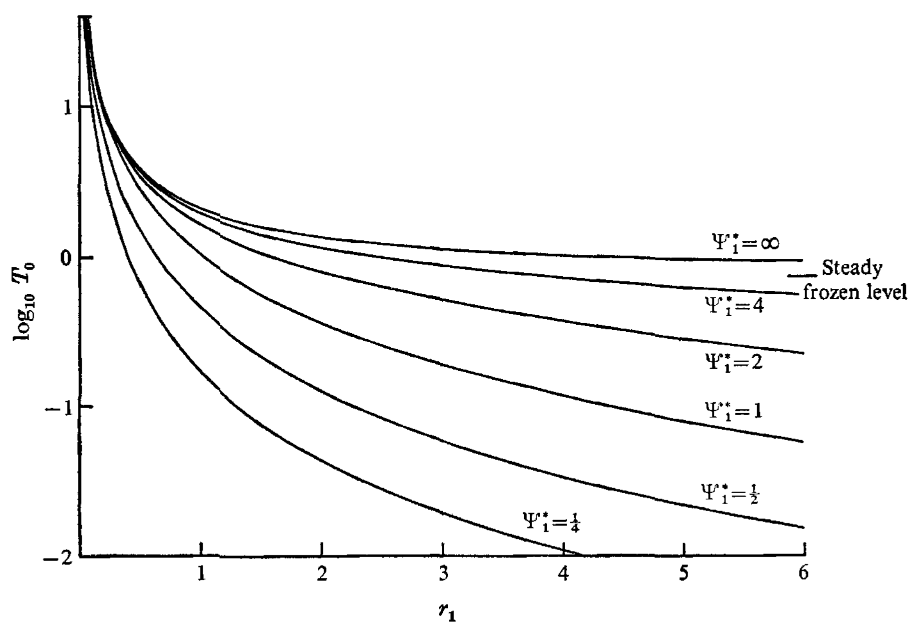

FIGURE 2. Computed variation of temperature (T_0) with scaled radial co-ordinate (r_1) for different particle path lines.

With these values the solution for temperature can be written,

$$T_0 = \left(\frac{k_2}{k_1^2 \Psi_1^{*2}} + \frac{1}{r_1}\right)^{\beta - \frac{2}{3}} r_1^{-2} \Psi_*\left(\beta, 2\beta + \tfrac{5}{3} + \frac{k_2}{k_1^3 \Psi_1^{*2}}; \frac{k_2}{k_1^2 \Psi_1^{*2}} + \frac{1}{r_1}\right). \tag{4.5}$$

For the purposes of computation the values $k_1 = k_2 = 1$ were chosen. With these particular values T_0 was calculated for different values of the scaled particle path line function, Ψ_1^*. Figure 2 gives a graphical representation of the results, for convenience $\log T_0$ is plotted against r_1. $\Psi_1^* = \infty$ corresponds to pure steady flow which results in a constant frozen temperature at infinity. For finite values of Ψ_1^*, we have a gradual breakaway from the 'freezing' type behaviour, and, from (3.37), the asymptotic variation of temperature for each Ψ_1^* as $r_1 \to \infty$ is given by

$$T_0 r_1^2 = \Psi_1^{*\frac{4}{3}-2\beta}\,\Psi_*(\beta, 2\beta+\tfrac{5}{3}+J;\, J)+O(1/r_1),$$

which is an augmented inviscid inverse square behaviour. An indication of the way in which the transition from the inviscid behaviour ($r \to 0$) to the final asymptotic variation ($r \to \infty$) occurs, is shown in figure 3.

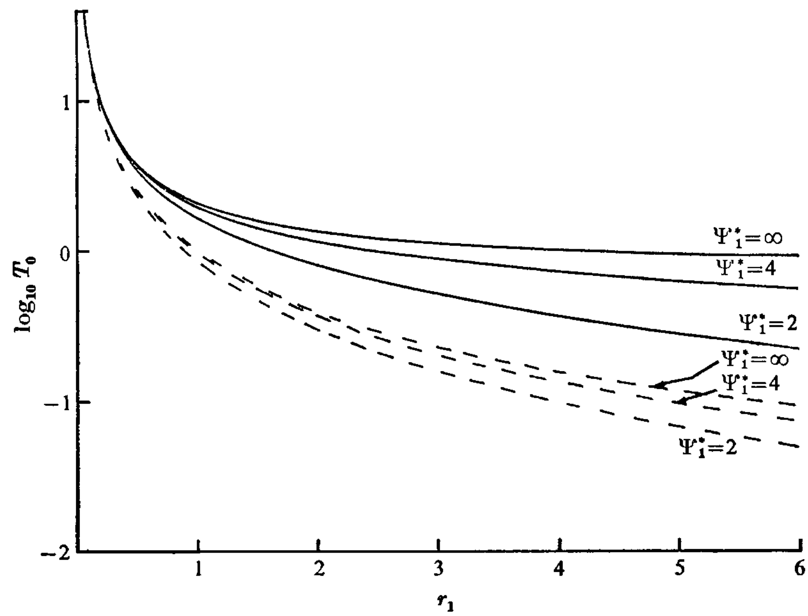

FIGURE 3. Illustration of the departure of temperature (T_0) from the curves predicted by inviscid theory for different values of Ψ_1^*. ———, inviscid flow.

5. Conclusions

The unsteady expansion of a monatomic gas into vacuum has been considered for spherical symmetry using the Maxwell molecule collision model. A solution of the inviscid equations of motion for large distances from the origin has been constructed for both spherical and cylindrical symmetry and has similar formulations for both geometries. In order to obtain these solutions the particle path function is employed, the major simplification being that the asymptotic velocity becomes solely dependent on this variable. The asymptotic solutions have the property, hitherto not discussed in the context of non-equilibrium expanding flows, that a steady state may be set up in some region of the flow field.

Confining the attention to flows with spherical symmetry the inviscid solution is treated as the zeroth-order term in an asymptotic expansion in powers of the inverse reference Knudsen number which is assumed small, and the behaviour of this expansion is discussed for large r. Three distinct regions of the flow field are revealed in the analysis. First, one in which the flow is mathematically equivalent to that of a fixed mass of gas considered by Grundy & Thomas (1969) in an earlier paper. Secondly, the region of prime interest, in which there is a transition between the first region and the steady state where the rate of change of the asymptotic velocity with particle path function is small. In this manner we approach the third distinct region; the steady state. Because of the generality of the method, an example, albeit of a somewhat artificial nature, is constructed in order to illustrate the essential properties of the solution in the second and third regions enumerated above.

Needless to say no attempt has been made to introduce molecular interactions other than that for Maxwell molecules. Mathematically, as far as the Boltzmann equation is concerned, the difficulties are immense. Even the moment equations associated with model equations such as the BGK type, appear to possess non-linear properties which make any analytic progress difficult.

One final point should be made concerning the flow near the zero density front predicted by the inviscid solution. As pointed out by Freeman & Grundy (1968) the present approximation technique does not result in any essential simplification of the Boltzmann equation in these regions. It is to be expected that similar difficulties would be encountered in the problems considered in this paper, and consequently no attempt at an analysis of the flow is made near the inviscid front.

A similar non-equilibrium analysis to the one followed in this paper can be pursued for the cylindrically symmetric case. However, the results obtained are basically different from those for spherical flow and will be dealt with in a later paper.

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